Catalan tree
&
Parity of some Sequences which are related to Catalan numbers

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Abstract

In this paper we determine the parity of some sequences which are related to Catalan numbers. Also we introduce a combinatorical object called, "Catalan tree", and discuss its properties.

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1 Introduction

Throughout this paper, for brevity, we represent the set of even counting numbers by the capital letter E, the set of odd counting numbers by the capital letter O, and the set of natural numbers, $\{1, 2, 3, 4, ...\}$, by $\mathbb{N}$.

In this paper we first define the term *Catalan tree* and then study its combinatorical properties. Later we explore the parity of some sequences which are related to Catalan numbers.

The Catalan numbers are an infinite sequence of integers 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ... They are defined by the following recurrence relation:

$$C_n = \sum_{i=1}^{n-1} C_i C_{n-i}, \ \ with\ C_0 = 0,\ C_1 = 1. \tag{1}$$

It also has the following explicit formula $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ and for large n, C_n behaves like $\frac{2^{2n}}{\sqrt{\pi n^3}}$, see [1].

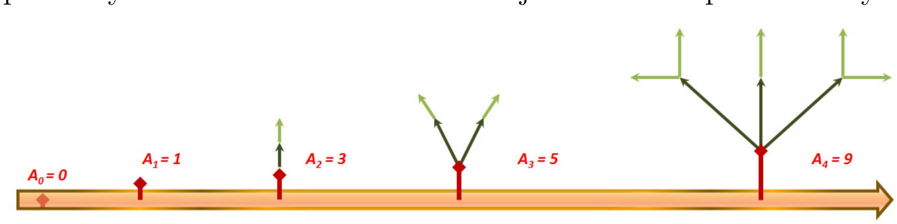
The Catalan number appears in many areas of mathematics just like the Fibonacci number. Specifically they are related in combinatorical settings such as trees, lattice paths, partitions, (see [4]) and even within propositional logic, (see [1]). Here we go one step further and define what a Catalan tree is.

Definition 1.1 *The nth* **Catalan tree**, *A_n, is a combinatorical object, characterized by one root, $(n-1)$ main-branches, and C_n sub-branches. Where each main-branch gives rise to a number of sub-branches, and the number of these sub-branches is determined by the additive partition of the corresponding Catalan number, as determined by the recurrence relation (1).*

The tree A_n can be represented symbolically as follows:

$$\begin{array}{rc} C_n \ \textit{sub-branches:} & (C_1C_{n-1}, C_2C_{n-2}, \ldots, C_{n-2}C_2, C_{n-1}C_1) \\ (n-1)\ \textit{main-branches:} & (1, 1, \ldots, 1, 1) \\ \textit{one root:} & (1) \end{array}$$

Note that the main-branches and the sub-branches exist iff $n > 1$. Here is an example: The Catalan tree A_4, has one root, (1), followed by 3 main-branches, $(1, 1, 1)$, and each main-branch gives rise to $(2, 1, 2)$ sub-branches respectively. Also this combinatorical object can be represented by a graph:



The diagram above shows the first five stages of the Catalan tree, A_n, where a_n is defined in Proposition 1.2.

Proposition 1.2 *For $n > 1$, let A_n denote the nth Catalan tree, and let a_n denote the number of components of A_n. Then*

$$a_n = C_n + n, \qquad \textit{with} \quad a_0 = 0, a_1 = 1.$$

Proof By definition there are 1 root, $(n-1)$ main branches and C_n sub-branches. Therefore $a_n = C_n + n$, for $n > 1$, and $a_0 = 0$, $a_1 = 1$. $\square$

Using Proposition 1.2, it is straightforward to calculate the values of a_n. The table below illustrates this up to $n = 10$.

n	0	1	2	3	4	5	6	7	8	9	10
a_n	0	1	3	5	9	19	48	139	437	1439	4872

Let $A(x)$, $C(x)$ and $N(x)$ denote the generating functions for a_n, C_n and n, respectively. Thus

$$A(x) = \sum_{n\geq 1} a_n x^n, \; C(x) = \sum_{n\geq 1} C_n x^n = \frac{1}{2}(1-\sqrt{1-4x}), \; N(x) = \sum_{n\geq 1} n x^n = \frac{x}{(1-x)^2}.$$

Proposition 1.3 *The generating function for the sequence $\{a_n\}_{n\geq 1}$ is given by*

$$A(x) = \frac{2x^2(2-x) + (1-x)^2(1-\sqrt{1-4x})}{2(1-x)^2}.$$

Corollary 1.4 *For $n > 1$, the explicit formula for a_n is given by,*

$$a_n = \frac{1}{n}\binom{2n-2}{n-1} + n.$$

The following result is a consequence of the asymptotic behavior of Catalan numbers.

Corollary 1.5 *For large n, we have the asymptotic formula*

$$a_n \sim \frac{2^{2n} + n^2\sqrt{\pi n}}{\sqrt{\pi n^3}}.$$

2 Parity of related sequences

In this section we determine the parity of the sequences that we have discussed in [1], and as well as the parity of the sequences which are related to the sequences in [1].

Note 2.1 *The following Lemma 2.2 has been proven by number of other authors, (see [2], (1986)), (see [4, page 330], (2004)) and (see [3], (2008)). But, they took n to be a Mersenne number, this is due to the fact that the Catalan numbers were shifted by one term forward in their work.*

From the Segner's recurrence relation, C_n can be expressed as a piecewise function, with respect to the parity of n, (see [4, page, 329]).

$$C_n = \begin{cases} 2(C_1C_{n-1} + C_2C_{n-2} + \ldots + C_{\frac{n-1}{2}}C_{\frac{n+1}{2}}) & \text{if } n \in O, \\ 2(C_1C_{n-1} + C_2C_{n-2} + \ldots + C_{\frac{n-2}{2}}C_{\frac{n+2}{2}}) + C_{\frac{n}{2}}^2 & \text{if } n \in E. \end{cases}$$

Lemma 2.2 (Parity of C_n)

$$C_n \in O \Longleftrightarrow n = 2^i, \text{ where } i \in \mathbb{N}.$$

Proof For $n \geq 2$

$$C_n \in O \Longleftrightarrow C_{\frac{n}{2}}^2 \in O \Longleftrightarrow C_{\frac{n}{2}} \in O \Longleftrightarrow n = 2^i \quad \forall i \in \mathbb{N}.$$

Note that $C_1 = 1 \in O$. $\square$

Corollary 2.3 (Parity of a_n)

$$a_n \in O \Longleftrightarrow n = 2^i \text{ or } n \in O,$$

(and $a_n \in E$ iff $n \in E$ and $n \neq 2^i$), for $i \in \mathbb{N}$.

Proof Let the symbols $\wedge$, and $\vee$ denote the connectives 'and' and 'or' respectively. Then

$$\begin{aligned} a_n = (C_n + n) \in O &\Longleftrightarrow (C_n \in O \wedge n \in E) \vee (C_n \in E \wedge n \in O) \\ &\Longleftrightarrow (n = 2^i \wedge n \in E) \vee (n \neq 2^i \wedge n \in O) \\ &\Longleftrightarrow (n = 2^i) \vee (n \in O). \end{aligned}$$

$\square$

Theorem 2.4 *Let f_n be the number of rows with the value "false" in the truth tables of all bracketed formulea with n distinct propositions $p_1, \dots, p_n$ connected by the binary connective of implication. Then in [1] we have shown that the following results are true:*

$$f_n = \sum_{i=1}^{n-1} (2^i C_i - f_i) f_{n-i}, \quad \textit{with} \;\; f_1 = 1 \tag{2}$$

and for large n, $f_n \sim \left(\frac{3-\sqrt{3}}{6}\right) \frac{2^{3n-2}}{\sqrt{\pi n^3}}$.

Using Theorem 2.4, we get the following triangular table. Where the left hand side column represents the sum of the corresponding row.

f_2:					1					
f_3:				1		3				
f_4:			4		3		12			
f_5:		19		12		12		61		
f_6:	104		57		48		61		344	

Theorem 2.5 (Parity of f_n) *The sequence $\{f_n\}_{n \geq 1}$ preserves the parity of C_n.*

Proof If an additive partition of f_n, (which is determined by the recurrence relation (2)), is odd, then it comes as a pair; i.e.

$$(2^i C_i - f_i) f_{n-i} \in O \iff f_i, f_{n-i} \in O \iff (2^{n-i} C_{n-i} - f_{n-i}) f_i \in O.$$

Hence, $\Big((2^i C_i - f_i) f_{n-i} + (2^{n-i} C_{n-i} - f_{n-i}) f_i\Big) \in E$.

Thus, f_n can be expressed as a piecewise function depending on the parity of n:

$$f_n = \begin{cases} \sum_{i=1}^{\frac{n-1}{2}} \left((2^i C_i - f_i) f_{n-i} + (2^{n-i} C_{n-i} - f_{n-i}) f_i\right) & \text{if } n \in O, \\ \left(\sum_{i=1}^{\frac{n-2}{2}} \left((2^i C_i - f_i) f_{n-i} + (2^{n-i} C_{n-i} - f_{n-i}) f_i\right)\right) + (2^{\frac{n}{2}} C_{\frac{n}{2}} - f_{\frac{n}{2}}) f_{\frac{n}{2}} & \text{if } n \in E. \end{cases}$$

Finally,

$$f_n \in O \iff (2^{\frac{n}{2}} C_{\frac{n}{2}} - f_{\frac{n}{2}}) f_{\frac{n}{2}} \in O \iff f_{\frac{n}{2}} \in O \iff n = 2^i, \;\; \forall i \in \mathbb{N}.$$

Note that $f_1 = 1 \in O$. $\square$

Theorem 2.6 *Let g_n be the total number of rows in all truth tables for bracketed implications with n distinct variables. Let t_n be the number of rows with the value "true" in the truth tables of all bracketed formulea with n distinct propositions $p_1, \ldots, p_n$ connected by the binary connective of implication. Then in [1] we have shown that the following results are true:*

$$t_n = g_n - f_n, \quad \text{with} \;\; t_0 = 0$$

and for large n, $t_n \sim \left(\frac{3+\sqrt{3}}{6}\right)\frac{2^{3n-2}}{\sqrt{\pi n^3}}$.

Proposition 2.7 (Parity of t_n) *The sequence $\{t_n\}_{n\geq 1}$ preserves the parity of C_n.*

Proof Since

$$t_n = g_n - f_n = 2^n C_n - f_n, \;\; \text{with } n \geq 1$$

The sequence $\{g_n\}_{n\geq 1}$ is always even, and the sequence $\{f_n\}_{n\geq 1}$ preserves the parity of C_n by Theorem 2.5. Therefore the sequence $\{t_n\}_{n\geq 1}$ preserves the parity of C_n. □

3 A fruitful tree

Definition 3.1 *The Catalan tree A_n is* **fruitful** *iff each sub-branch of A_n has fruits. We denote this new tree by $A_n(\mu_i)$, where $\{\mu_i\}_{i\geq 1}$ is the corresponding fruit sequence.*

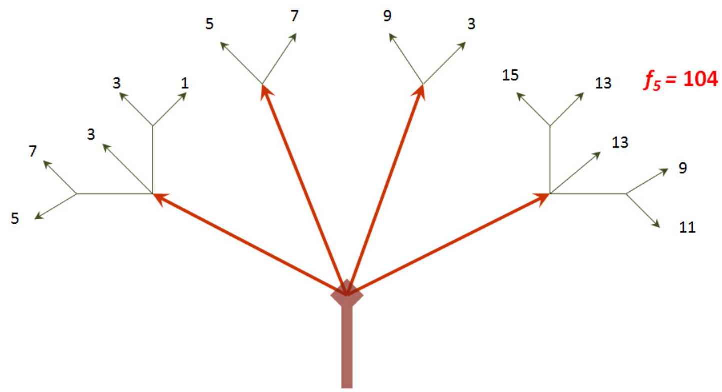
Example 3.2 *Let $\{f_n\}_{n\geq 1}$ be the corresponding fruit sequence for the Catalan tree A_n. Then $A_n(f_n)$ has the following symbolic representation,*

$$\begin{gathered}((2^1C_1 - f_1)f_{n-1}, (2^2C_2 - f_2)f_{n-2}, \ldots, (2^{n-2}C_{n-2} - f_{n-2})C_2, (2^{n-1}C_{n-1} - f_{n-1})f_1) \\ (C_1C_{n-1}, C_2C_{n-2}, \ldots, C_{n-2}C_2, C_{n-1}C_1) \\ (1, 1, \ldots, 1, 1) \\ (1).\end{gathered}$$

Example 3.3 *More concretely, $A_5(f_n)$ has the following symbolic representation:*

$$\begin{gathered}((5, 7, 3, 3, 1), (5, 7), (9, 3), (15, 13, 13, 9, 11)) \\ (5, 2, 2, 5) \\ (1, 1, 1, 1) \\ (1).\end{gathered}$$

The diagram below shows $A_5(f_n)$ as a graph:



Example 3.4 *Let $\{t_n\}_{n\geq 1}$ be the corresponding fruit sequence for the Catalan tree A_n. Then $A_n(t_n)$ has the following symbolic representation,*

$$(2^n - ((2^1C_1 - f_1)f_{n-1}), \ldots, (2^n - (2^{n-1}C_{n-1} - f_{n-1})f_1))$$
$$(C_1C_{n-1}, C_2C_{n-2}, \ldots, C_{n-2}C_2, C_{n-1}C_1)$$
$$(1, 1, \ldots, 1, 1)$$
$$(1).$$

Proposition 3.5 *For $n > 1$, let $a_n(f_n)$ and $a_n(t_n)$ be the total number of components of the fruitful trees $A_n(f_n)$ and $A_n(t_n)$ respectively. Then*

$$a_n(f_n) = f_n + C_n + n, \quad \textit{and} \quad a_n(t_n) = t_n + C_n + n.$$

Using Proposition 3.5, it is straightforward to calculate the values of $a_n(f_n)$, and $a_n(t_n)$. The table below illustrates this up to $n = 10$.

n	0	1	2	3	4	5	6	7	8	9	10
$a_n(f_n)$	0	2	4	9	28	123	662	3955	25032	164335	1106794
$a_n(t_n)$	0	2	6	17	70	363	2122	13219	85666	570703	3881638

Corollary 3.6 *For $n \geq 1$, $a_n(f_n)$, and $a_n(t_n)$ are odd iff $n \in O$.*

Proof Since,

$$f_n, t_n \in O \Longleftrightarrow n = 2^i, \ \textit{and}\ a_n = (C_n + n) \in O \Longleftrightarrow n = 2^i \ \textit{or}\ n \in O$$

then $a_n(f_n), a_n(t_n) \in O \Longleftrightarrow n \in O.$ $\square$

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